

Sturm comparison theorems via Picone-type inequalities for some nonlinear elliptic type equations with damped terms

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Abstract. In this paper, we establish a Picone-type inequality for a class of some nonlinear elliptic type equations with damped terms, and obtain Sturmian comparison theorems using the Picone-type inequality. As an application by using comparison theorem oscillation result and Wirtinger-type inequality are given.

Keywords: Picone-type inequality, elliptic equations, Sobolev space, half-linear equations, oscillation criteria, Wirtinger-type inequality.

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1 Introduction

Since the pioneering work of Sturm [27] in 1836, Sturmian comparison theorems have been derived for differential equations of various types. In order to obtain Sturmian comparison theorems for ordinary differential equations of second order, Picone [25] established an identity, known as the Picone identity. In the latter years, Jaroš and Kusano [15] derived a Picone-type identity for half-linear differential equations of second order. They also developed Sturmian theory for both forced and unforced half-linear and quasilinear equations based on this identity. Since Picone identities play an important role in the study of qualitative theory of differential equations, establishing Picone identities has become a popular research topic. We refer the reader to Kreith [20, 21], Swanson [28, 29] for Picone identities and Sturmian comparison theorems for linear elliptic equations and to Allegretto [3], Allegretto and Huang [4, 5], Bognár and Došlý [9], Dunninger [12], Kusano, Jaroš and Yoshida [22], Yoshida [32, 31, 30] for Picone identities, Sturmian comparison and/or oscillation theorems for half-linear elliptic equations. In particular, we mention the paper [12] by Dunninger which seems to be the first paper dealing with Sturmian comparison theorems for half-linear elliptic equations.

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Recently, Yoshida [35] established Sturmian comparison and oscillation theorems for quasi-linear undamped elliptic operators with mixed nonlinearities in the following forms,

$$\begin{aligned}\ell(u) &:= \sum_{k=1}^m \nabla \cdot \left(a_k(x) |\sqrt{a_k(x)} \nabla u|^{\alpha-1} \nabla u \right) + c(x) |u|^{\alpha-1} u, \\ L(v) &:= \sum_{k=1}^m \nabla \cdot \left(A_k(x) |\sqrt{A_k(x)} \nabla v|^{\alpha-1} \nabla v \right) + g(x, v)\end{aligned}$$

where $a_k(x), A_k(x)$ are matrices and

$$g(x, v) = C(x) |v|^{\alpha-1} v + \sum_{i=1}^{\ell} D_i(x) |v|^{\beta_i-1} v + \sum_{j=1}^m E_j(x) |v|^{\gamma_j-1} v.$$

Most of the work in the literature deals with the Sturmian comparison results for elliptic equations that contain undamped terms. In this paper, we establish Sturmian comparison theorems for a pair of damped elliptic operators p and P defined by

$$p(u) := \nabla \cdot (a(x) |\nabla u|^{\alpha-1} \nabla u) + (\alpha + 1) |\nabla u|^{\alpha-1} b(x) \cdot \nabla u + c(x) |u|^{\alpha-1} u, \quad (1.1)$$

$$P(v) := \nabla \cdot (A(x) |\nabla v|^{\alpha-1} \nabla v) + (\alpha + 1) |\nabla v|^{\alpha-1} B(x) \cdot \nabla v + g(x, v), \quad (1.2)$$

where $|\cdot|$ denotes the Euclidean length, $\alpha > 0$ is a constant, $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})^T$, (the superscript T denotes the transpose). It is assumed that $\beta_i > \alpha > \gamma_j > 0$ ($i = 1, 2, \dots, \ell$; $j = 1, 2, \dots, m$). To the best of our knowledge, damped elliptic operators such as $p(u)$ and $P(v)$ defined as above have not been studied.

Note that the principal part of (1.1) and (1.2) are reduced to the p -Laplacian $\nabla \cdot (|\nabla u|^{p-2} \nabla u)$, ($p = \alpha + 1$). We know that a variety of physical phenomena are modeled by equations involving the p -Laplacian [2, 7, 8, 23, 24, 26]. We refer the reader to Diaz [11] for detailed references on physical background of the p -Laplacian.

We organize this paper as follows. In Section 2, we establish a Picone-type inequality. In Section 3, we present comparison results for the equations $p(u) = 0$ and $P(v) = 0$ and in Section 4, as an application we conclude some oscillation results and give a Wirtinger-type inequality.

2 Picone-type inequalities

In this section, we establish a Picone-type inequality for the coupled operators p and P defined by (1.1) and (1.2) respectively. Let G be a bounded domain in R^n with piecewise smooth boundary ∂G , and assume that $a(x) \in C(\bar{G}, R^+)$, $A(x) \in C(\bar{G}, R^+)$, $b(x) \in C(\bar{G}, R^n)$, $B(x) \in C(\bar{G}, R^n)$, $c(x) \in C(\bar{G}, R)$, $C(x) \in C(\bar{G}, R)$, $D_i(x) \in C(\bar{G}, [0, \infty))$, $E_j(x) \in C(\bar{G}, [0, \infty))$, ($i = 1, 2, \dots, \ell$; $j = 1, 2, \dots, m$).

The domain $\mathcal{D}_p(G)$ of p is defined to be the set of all functions u of class $C^1(\bar{G}, R)$ with the property that $a(x) |\nabla u|^{\alpha-1} \nabla u \in C^1(G, R^n) \cap C(\bar{G}, R^n)$. The domain $\mathcal{D}_P(G)$ of P is defined similarly.

Let $N = \min\{\ell, m\}$ and

$$H(\beta, \alpha, \gamma; D(x), E(x)) = \frac{\beta - \gamma}{\alpha - \gamma} \left(\frac{\beta - \alpha}{\alpha - \gamma} \right)^{\frac{\alpha - \beta}{\beta - \gamma}} (D(x))^{\frac{\alpha - \gamma}{\beta - \gamma}} (E(x))^{\frac{\beta - \alpha}{\beta - \gamma}}.$$

We will need the following lemmas, in order to prove our results.

Lemma 2.1 ([22, Lemma 2.1]). *The inequality*

$$|X|^{\alpha+1} + \alpha|Y|^{\alpha+1} - (\alpha+1)|Y|^{\alpha-1}X \cdot Y \geq 0.$$

is valid for any $X \in R^n$ and $Y \in R^n$, where the equality holds if and only if $X = Y$.

Lemma 2.2 ([32, Lemma 8.3.2]). *Let $F(x) \in C(G, R^+)$ satisfy $F(x) > \alpha > 0$. Then the inequality*

$$|\nabla u - uw(x)|^{\alpha+1} \leq \frac{F(x)}{F(x) - \alpha} |\nabla u|^{\alpha+1} + \frac{|F(x)w(x)|^{\alpha+1}}{F(x) - \alpha} |u|^{\alpha+1}$$

holds for any function $u \in C^1(G, R)$ and any n -vector function $w(x) \in C(G, R^n)$.

Theorem 2.3 (Picone-type inequality). *Let $F(x) \in C(G, R^+)$ satisfying $F(x) > \alpha$. If $u \in \mathcal{D}_p(G)$, $v \in \mathcal{D}_p(G)$ and $v \neq 0$ in G (that is, v has no zero in G), then the following Picone-type inequality holds:*

$$\begin{aligned} & \nabla \cdot \left(\frac{u}{\varphi(v)} [\varphi(v)a(x)|\nabla u|^{\alpha-1}\nabla u - \varphi(u)A(x)|\nabla v|^{\alpha-1}\nabla v] \right) \\ & \geq \left(a(x) - \alpha|b(x)| - A(x)\frac{F(x)}{F(x) - \alpha} \right) |\nabla u|^{\alpha+1} \\ & \quad + \left(C_1(x) - c(x) - |b(x)| - A(x)\frac{|F(x)B(x)/A(x)|^{\alpha+1}}{F(x) - \alpha} \right) |u|^{\alpha+1} \\ & \quad + A(x) \left[\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{uB(x)}{A(x)} \right) \cdot \Phi \left(\frac{u}{v} \nabla v \right) \right] \\ & \quad + \frac{u}{\varphi(v)} (\varphi(v)p(u) - \varphi(u)P(v)), \end{aligned} \tag{2.1}$$

where $\varphi(s) = |s|^{\alpha-1}s$, $s \in R$, $\Phi(\xi) = |\xi|^{\alpha-1}\xi$, $\xi \in R^n$ and

$$C_1(x) = C(x) + \sum_{i=1}^N H(\beta_i, \alpha_i, \gamma_i; D_i(x), E_i(x)).$$

Proof. We easily see that

$$\begin{aligned} \nabla \cdot \left(ua(x)|\nabla u|^{\alpha-1}\nabla u \right) &= a(x)|\nabla u|^{\alpha+1} - c(x)|u|^{\alpha+1} \\ &\quad + up(u) - (\alpha+1)ub(x) \cdot \Phi(\nabla u). \end{aligned} \tag{2.2}$$

We observe that the following identity holds:

$$\begin{aligned} & -\nabla \cdot \left(u\varphi(u)\frac{A(x)|\nabla v|^{\alpha-1}\nabla v}{\varphi(v)} \right) \\ &= -A(x) \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} \\ &\quad + A(x) \left[\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{uB(x)}{A(x)} \right) \cdot \Phi \left(\frac{u}{v} \nabla v \right) \right] \\ &\quad + \frac{u\varphi(u)}{\varphi(v)} g(x, v) - \frac{u\varphi(u)}{\varphi(v)} P(v). \end{aligned} \tag{2.3}$$

We combine (2.2) with (2.3) to obtain the following:

$$\begin{aligned}
& \nabla \cdot \left(\frac{u}{\varphi(v)} \left[\varphi(v)a(x)|\nabla u|^{\alpha-1}\nabla u - \varphi(u)A(x)|\nabla v|^{\alpha-1}\nabla v \right] \right) \\
&= a(x)|\nabla u|^{\alpha+1} - c(x)|u|^{\alpha+1} - (\alpha+1)ub(x) \cdot \Phi(\nabla u) - A(x) \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} \\
&+ A(x) \left[\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{uB(x)}{A(x)} \right) \cdot \Phi \left(\frac{u}{v} \nabla v \right) \right] \\
&+ \frac{u\varphi(u)}{\varphi(v)}g(x,v) + \frac{u}{\varphi(v)}[\varphi(v)p(u) - \varphi(u)P(v)].
\end{aligned} \tag{2.4}$$

Using Young's inequality we have,

$$\begin{aligned}
\frac{u\varphi(u)}{\varphi(v)}g(x,v) &\geq C(x)|u|^{\alpha+1} + \left(\sum_{i=1}^N H(\beta_i, \alpha_i, \gamma_i; D_i(x), E_i(x)) \right) |u|^{\alpha+1} \\
&= C_1(x)|u|^{\alpha+1}
\end{aligned} \tag{2.5}$$

and

$$(\alpha+1)ub(x) \cdot \Phi(\nabla u) \leq |b(x)| \left(|u|^{\alpha+1} + \alpha |\nabla u|^{\alpha+1} \right). \tag{2.6}$$

From Lemma 2.2, we can write

$$\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} \leq \frac{F(x)}{F(x) - \alpha} |\nabla u|^{\alpha+1} + \frac{\left| F(x) \frac{B(x)}{A(x)} \right|^{\alpha+1}}{F(x) - \alpha} |u|^{\alpha+1}. \tag{2.7}$$

We combine (2.5)–(2.7) with (2.4) to obtain the desired inequality (2.1). \square

Theorem 2.4. *If $v \in \mathcal{D}_P(G)$, and $v \neq 0$ in G , then the following inequality holds for any $u \in C^1(G, R)$:*

$$\begin{aligned}
& -\nabla \cdot \left(\frac{u\varphi(u)}{\varphi(v)} A(x) |\nabla v|^{\alpha-1} \nabla v \right) \\
& \geq -A(x) \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} \\
& + A(x) \left[\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{uB(x)}{A(x)} \right) \cdot \Phi \left(\frac{u}{v} \nabla v \right) \right] \\
& + C_1(x)|u|^{\alpha+1} - \frac{u\varphi(u)}{\varphi(v)}P(v),
\end{aligned} \tag{2.8}$$

where $\varphi(s)$, $\Phi(\xi)$ and $C_1(x)$ are defined as in Theorem 2.3.

Proof. Combining (2.3) with (2.5) yields the desired inequality (2.8). \square

3 Sturmian comparison theorems

In this section we present some Sturmian comparison results on the basis of the Picone-type inequality obtained in Section 2.

Theorem 3.1 (Sturmian comparison theorem). *Let $F(x) \in C(G, R^+)$ satisfy $F(x) > \alpha$. If there exists a nontrivial solution $u \in \mathcal{D}_p(G)$ of $p(u) = 0$ such that $u = 0$ on ∂G and*

$$V(u) := \int_G \left[\left(a(x) - \alpha |b(x)| - A(x) \frac{F(x)}{F(x) - \alpha} \right) |\nabla u|^{\alpha+1} + \left(C_1(x) - c(x) - |b(x)| - A(x) \frac{|F(x)B(x)/A(x)|^{\alpha+1}}{F(x) - \alpha} \right) |u|^{\alpha+1} \right] dx \geq 0 \quad (3.1)$$

then every solution $v \in \mathcal{D}_p(G)$ of $P(v) = 0$ must vanish at some point of \bar{G} .

Proof. Suppose that, contrary to our claim there exists a solution $v \in \mathcal{D}_p(G)$ of $P(v) = 0$ satisfying $v \neq 0$ on \bar{G} . We integrate (2.1) over G and then apply the divergence theorem to obtain

$$0 \geq V(u) + \int_G A(x) \left[\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha + 1) \left(\nabla u - \frac{uB(x)}{A(x)} \right) \cdot \Phi \left(\frac{u}{v} \nabla v \right) \right] dx \geq 0 \quad (3.2)$$

and therefore

$$\int_G A(x) \left[\left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha + 1) \left(\nabla u - \frac{uB(x)}{A(x)} \right) \cdot \Phi \left(\frac{u}{v} \nabla v \right) \right] dx = 0. \quad (3.3)$$

From Lemma 2.1, we see that

$$\nabla u - \frac{uB(x)}{A(x)} \equiv \frac{u}{v} \nabla v \text{ or } \nabla \left(\frac{u}{v} \right) - \frac{B(x)}{A(x)} \frac{u}{v} \equiv 0 \text{ in } G, \quad (3.4)$$

then it follows from a result of Jaroš, Kusano and Yoshida [17] that

$$\frac{u}{v} = C_0 e^{\alpha(x)} \text{ on } \bar{G} \quad (3.5)$$

for some constant C_0 and some continuous function $\alpha(x)$. Since $u = 0$ on ∂G , we see that $C_0 = 0$, which contradicts the fact that u is nontrivial. The proof is complete. \square

Corollary 3.2. *Let $F(x) \in C(G, R^+)$ satisfy $F(x) > \alpha$. Assume that*

$$a(x) \geq \alpha |b(x)| + A(x) \frac{F(x)}{F(x) - \alpha} \quad (3.6)$$

and

$$C_1(x) \geq c(x) + |b(x)| + A(x) \frac{\left| F(x) \frac{B(x)}{A(x)} \right|^{\alpha+1}}{F(x) - \alpha} \quad (3.7)$$

in G . If there exists a nontrivial solution $u \in \mathcal{D}_p(G)$ of $p(u) = 0$ such that $u = 0$ on ∂G , then every solution $v \in \mathcal{D}_p(G)$ of $P(v) = 0$ must vanish at some point of \bar{G} .

Theorem 3.3. *If there exists a nontrivial function $u \in C^1(\bar{G}, R)$ such that $u = 0$ on ∂G and*

$$M(u) := \int_G \left\{ A(x) \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} - C_1(x)|u|^{\alpha+1} \right\} dx \leq 0 \quad (3.8)$$

then every solution $v \in \mathcal{D}_P(G)$ of $P(v) = 0$ must vanish at some point of G unless $u = C_0 e^{\alpha(x)} v$, where $C_0 \neq 0$ is a constant and $\nabla \alpha(x) = \frac{B(x)}{A(x)}$ in G .

Proof. Suppose that there exists a solution $v \in \mathcal{D}_P(G)$ of $P(v) = 0$ satisfying $v \neq 0$ in G . Since $\partial G \in C^1$, $u \in C^1(\bar{G}, R)$ and $u = 0$ on ∂G , we find that u belongs to the Sobolev space $W_0^{1, \alpha+1}(G)$ which is the closure in the norm

$$\|w\| := \left(\int_G \left[|w|^{\alpha+1} + |\nabla w|^{\alpha+1} \right] dx \right)^{\frac{1}{\alpha+1}} \quad (3.9)$$

of the class $C_0^\infty(G)$ of infinitely differentiable functions with compact supports in G [1, 13]. Then there is a sequence u_k of functions in $C_0^\infty(G)$ converging to u in the norm (3.9). Integrating (2.8) with $u = u_k$ over G , then applying the divergence theorem, we have

$$\begin{aligned} M(u_k) \geq \int_G A(x) \left[\left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u_k}{v} \nabla v \right|^{\alpha+1} \right. \\ \left. - (\alpha+1) \left(\nabla u_k - \frac{u_k B(x)}{A(x)} \right) \cdot \Phi \left(\frac{u_k}{v} \nabla v \right) \right] dx \geq 0. \end{aligned} \quad (3.10)$$

We first claim that $\lim_{k \rightarrow +\infty} M(u_k) = M(u) = 0$. Since $A(x)$, $C(x)$, $D(x)$ and $E(x)$ are bounded on \bar{G} , there exists a constant $K_1 > 0$ such that

$$A(x) \leq K_1 \quad \text{and} \quad |C_1(x)| \leq K_1. \quad (3.11)$$

It is easy to check that

$$\begin{aligned} |M(u_k) - M(u)| \leq K_1 \int_G \left| \left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} - \left| \nabla u - \frac{u B(x)}{A(x)} \right|^{\alpha+1} \right| dx \\ + K_1 \int_G \left| |u_k|^{\alpha+1} - |u|^{\alpha+1} \right| dx. \end{aligned} \quad (3.12)$$

From the mean value theorem we see that

$$\begin{aligned} & \left| \left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} - \left| \nabla u - \frac{u B(x)}{A(x)} \right|^{\alpha+1} \right| \\ & \leq (\alpha+1) \left(\left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right| + \left| \nabla u - \frac{u B(x)}{A(x)} \right| \right)^\alpha \left| \nabla(u_k - u) + \frac{B(x)}{A(x)}(u_k - u) \right| \\ & \leq (\alpha+1) \left(|\nabla u_k| + |\nabla u| + \frac{|B(x)|}{A(x)}|u_k| + \frac{|B(x)|}{A(x)}|u| \right)^\alpha \left(|\nabla(u_k - u)| + \frac{|B(x)|}{A(x)}|u_k - u| \right). \end{aligned}$$

Since also $B(x)$ is bounded on \bar{G} , then there is a constant K_2 such that $\frac{|B(x)|}{A(x)} \leq K_2$ on \bar{G} . Let us take $K_3 = \max\{1, K_2\}$. From the above inequality we have

$$\begin{aligned} & \left| \left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} - \left| \nabla u - \frac{u B(x)}{A(x)} \right|^{\alpha+1} \right| \\ & \leq (\alpha+1) K_3^{\alpha+1} (|\nabla u_k| + |\nabla u| + |u_k| + |u|)^\alpha (|\nabla(u_k - u)| + |u_k - u|). \end{aligned} \quad (3.13)$$

Using (3.13) and applying Hölder's inequality, we get

$$\begin{aligned}
& \int_G \left| \left| \nabla u_k - \frac{u_k B(x)}{A(x)} \right|^{\alpha+1} - \left| \nabla u - \frac{u B(x)}{A(x)} \right|^{\alpha+1} \right| dx \\
& \leq (\alpha+1) K_3^{\alpha+1} \left(\int_G (|\nabla u_k| + |\nabla u| + |u_k| + |u|)^{\alpha+1} dx \right)^{\frac{\alpha}{\alpha+1}} \\
& \quad \times \left(\int_G (|\nabla(u_k - u)| + |u_k - u|)^{\alpha+1} dx \right)^{\frac{1}{\alpha+1}} \\
& \leq (\alpha+1) K_3^{\alpha+1} \|u_k - u\| (\|u_k\| + \|u\|)^\alpha.
\end{aligned} \tag{3.14}$$

Similarly, we obtain

$$\int_G \left| |u_k|^{\alpha+1} - |u|^{\alpha+1} \right| dx \leq (\alpha+1) (\|u_k\| + \|u\|)^\alpha \|u_k - u\|. \tag{3.15}$$

Combining (3.12), (3.14) and (3.15), we have

$$|M(u_k) - M(u)| \leq K_4 (\|u_k\| + \|u\|)^\alpha \|u_k - u\| \tag{3.16}$$

for some positive constant $K_4 = K_4(K_1, K_2, K_3)$ and so that $\lim_{k \rightarrow +\infty} M(u_k) = M(u)$. We get from (3.10) that $M(u) \geq 0$ which together with (3.8) implies $M(u) = 0$.

Let \mathcal{B} be an arbitrary ball with $\bar{\mathcal{B}} \subset G$ and define

$$\begin{aligned}
Q_{\mathcal{B}}(w) := \int_{\mathcal{B}} A(x) & \left[\left| \nabla w - \frac{w B(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{w}{v} \nabla v \right|^{\alpha+1} \right. \\
& \left. - (\alpha+1) \left(\nabla w - \frac{w B(x)}{A(x)} \right) \cdot \Phi \left(\frac{w}{v} \nabla v \right) \right] dx \tag{3.17}
\end{aligned}$$

for $w \in C^1(G, \mathbb{R})$.

It is easy to check that

$$0 \leq Q_{\mathcal{B}}(u_k) \leq Q_G(u_k) \leq M(u_k), \tag{3.18}$$

where $Q_G(u_k)$ denotes the right-hand side of (3.17) with $w = u_k$ and with \mathcal{B} replaced by G .

A simple calculation yields

$$\begin{aligned}
|Q_{\mathcal{B}}(u_k) - Q_{\mathcal{B}}(u)| & \leq K_5 (\|u_k\|_{\mathcal{B}} + \|u\|_{\mathcal{B}})^\alpha \|u_k - u\|_{\mathcal{B}} + K_6 (\|u_k\|_{\mathcal{B}})^\alpha \|u_k - u\|_{\mathcal{B}} \\
& \quad + K_7 \|\varphi(u_k) - \varphi(u)\|_{L^q_{(\mathcal{B})}} \|u\|_{\mathcal{B}},
\end{aligned} \tag{3.19}$$

where $q = \frac{\alpha+1}{\alpha}$, the constants K_5, K_6 and K_7 are independent of k and the subscript \mathcal{B} indicates the integrals involved in the norm (3.9) are to be taken over \mathcal{B} instead of G . It is known that the Nemitski operator $\varphi: L^{\alpha+1}(G) \rightarrow L^q(G)$ is continuous [6] and it is clear that $\|u_k - u\|_{\mathcal{B}} \rightarrow 0$ as $\|u_k - u\|_G \rightarrow 0$.

Therefore, letting $k \rightarrow \infty$ in (3.18), we find that $Q_{\mathcal{B}}(u) = 0$. Since $A(x) > 0$ in \mathcal{B} , it follows that

$$\left[\left| \nabla u - \frac{u B(x)}{A(x)} \right|^{\alpha+1} + \alpha \left| \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha+1) \left(\nabla u - \frac{u B(x)}{A(x)} \right) \cdot \Phi \left(\frac{u}{v} \nabla v \right) \right] \equiv 0 \text{ in } \mathcal{B}, \tag{3.20}$$

from which Lemma 2.1 implies that

$$\nabla u - \frac{uB(x)}{A(x)} \equiv \frac{u}{v} \nabla v \text{ or } \nabla \left(\frac{u}{v} \right) - \frac{B(x)}{A(x)} \frac{u}{v} \equiv 0 \text{ in } \mathcal{B}.$$

Hence we observe that $\frac{u}{v} = C_0 e^{\alpha(x)}$ in \mathcal{B} for some constant C_0 and some continuous function $\alpha(x)$ as in the proof of Theorem 3.1. Since \mathcal{B} is an arbitrary ball with $\bar{\mathcal{B}} \subset G$, we conclude that $\frac{u}{v} = C_0 e^{\alpha(x)}$ in G where $C_0 \neq 0$. \square

Corollary 3.4 (Sturmian comparison theorem). *Let $F(x) \in C(G, R^+)$ satisfy $F(x) > \alpha$. If there exists a nontrivial solution $u \in \mathcal{D}_p(G)$ of $p(u) = 0$ for which $u = 0$ on ∂G and (3.1) hold, then every solution $v \in D_p(G)$ of $P(v) = 0$ must vanish at some point of G unless $u = C_0 e^{\alpha(x)} v$, where $C_0 \neq 0$ is a constant and $\nabla \alpha(x) = \frac{B(x)}{A(x)}$ in G .*

Proof. By using (2.2), (2.6), (2.7), (3.8) and Corollary 3.2 we obtain

$$M(u) \leq \int_G \left[\nabla \cdot \left(u a(x) |\nabla u|^{\alpha-1} \nabla u \right) - u p(u) \right] dx = 0.$$

Hence the result follows from Theorem 3.3. \square

Remark 3.5. When we take $\alpha = 1$, $b(x) \equiv B(x) \equiv 0$ and $D_i(x) \equiv E_i(x) \equiv 0$, ($i = 1, 2, \dots, \ell$, $j = 1, 2, \dots, m$) that is, in the linear elliptic equation case, and $b(x) \equiv B(x) \equiv 0$ and $D_i(x) \equiv E_i(x) \equiv 0$, ($i = 1, 2, \dots, \ell$, $j = 1, 2, \dots, m$) that is, in the half-linear elliptic equation case, our results cannot be reduced to the well-known results. Hence our results are indeed a partial extension of the results that are given in the literature. Improvement of our results is left as an open problem to the researchers.

4 Applications

Let Ω be an exterior domain in R^n , that is, $\Omega \supset \{x \in R^n : |x| \geq r_0\}$ for some $r_0 > 0$. We consider the following equations:

$$p(u) = 0 \text{ in } \Omega \tag{4.1}$$

and

$$P(v) = 0 \text{ in } \Omega \tag{4.2}$$

where the operators p and P are defined in Section 1 and $a, A \in C(\Omega, R^+)$, $b, B \in C(\Omega, R^n)$, $c, C \in C(\Omega, R)$, $D_i, E_j \in C(\Omega, [0, \infty))$, ($i = 1, 2, \dots, \ell$; $j = 1, 2, \dots, m$).

The domain $\mathcal{D}_p(\Omega)$ of p is defined to be the set of all functions u of class $C^1(\Omega, R)$ with the property that $a(x) |\nabla u|^{\alpha-1} \nabla u \in C^1(\Omega, R^n)$. The domain $\mathcal{D}_P(\Omega)$ of P is defined similarly.

A solution $u \in \mathcal{D}_p(\Omega)$ of (4.1) (or $v \in \mathcal{D}_P(\Omega)$ of (4.2)) is said to be oscillatory in Ω if it has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap \{x \in R^n : |x| > r\}.$$

A bounded domain G with $\bar{G} \subset \Omega$ is said to be a nodal domain for the equation (4.1), if there exists a nontrivial function $u \in \mathcal{D}_p(G)$ such that $p(u) = 0$ in G and $u = 0$ on ∂G . The equation (4.1) is called nodally oscillatory in Ω , if (4.1) has a nodal domain contained in Ω_r for any $r > 0$.

Theorem 4.1. Let $F(x) \in C(G, \mathbb{R}^+)$ satisfy $F(x) > \alpha$. Assume that

$$a(x) \geq \alpha |b(x)| + A(x) \frac{F(x)}{F(x) - \alpha} \quad (4.3)$$

and

$$C_1(x) \geq c(x) + |b(x)| + A(x) \frac{\left| F(x) \frac{B(x)}{A(x)} \right|^{\alpha+1}}{F(x) - \alpha} \quad (4.4)$$

in Ω . If (4.1) is nodally oscillatory in Ω , then every solution $v \in \mathcal{D}_p(G)$ of (4.2) is oscillatory in Ω .

Proof. Since (4.1) is nodally oscillatory in Ω , there exist a nodal domain $G \subset \Omega_r$ for any $r > 0$, and hence there exists a nontrivial function $u \in D_p(G)$ such that $p(u) = 0$ in G and $u = 0$ on ∂G . The conditions (4.3) and (4.4) ensures that $V(u) \geq 0$ is satisfied. From Corollary 3.2 it follows that every solution $v \in \mathcal{D}_p(\Omega)$ of (4.2) vanishes at some point of \bar{G} , that is, v must have a zero in Ω_r for any $r > 0$. This implies that v is oscillatory in Ω . \square

The following is an immediate consequence of Theorem 4.1 by choosing $F(x) = \alpha + 1$, $b(x) \equiv B(x) \equiv 0$ and $m = 1$.

Corollary 4.2. If the equation

$$\nabla \cdot \left(a(x) |\nabla u|^{\alpha-1} \nabla u \right) + \left\{ C(x) + \frac{\beta - \gamma}{\alpha - \gamma} \left(\frac{\beta - \alpha}{\alpha - \gamma} \right)^{\frac{\alpha-\beta}{\beta-\gamma}} (D(x))^{\frac{\alpha-\gamma}{\beta-\gamma}} (E(x))^{\frac{\beta-\alpha}{\beta-\gamma}} \right\} |u|^{\alpha-1} u = 0 \quad (4.5)$$

is nodally oscillatory in Ω , then every solution $v \in \mathcal{D}_p(\Omega)$ of the equation

$$\nabla \cdot \left(a(x) |\nabla v|^{\alpha-1} \nabla v \right) + \frac{1}{\alpha + 1} g(x, v) = 0$$

is oscillatory in Ω , where $D_1(x) \equiv D(x)$, $E_1(x) \equiv E(x)$, $\alpha_1 \equiv \alpha$, $\gamma_1 \equiv \gamma$.

Various criteria for nodal oscillation can be found in [32]. For example for linear elliptic equations of the form

$$\Delta u + c(x)u = 0, \quad x \in \mathbb{R}^2, \quad (4.6)$$

$c(x)$ being a continuous function in \mathbb{R}^2 , have been given by Kreith and Travis [19]. They showed that (4.6) is nodally oscillatory if

$$\int_{\mathbb{R}^2} c(x) dx = \infty.$$

Applying this result to the equation (4.5) with $\alpha = 1$, $a(x) \equiv 1$ we have the following result.

Corollary 4.3. If one of the following holds; either

$$\int_{\mathbb{R}^2} C(x) dx = \infty$$

or

$$\int_{\mathbb{R}^2} C(x) dx \text{ exists, and } \int_{\mathbb{R}^2} (D(x))^{\frac{1-\gamma}{\beta-\gamma}} (E(x))^{\frac{\beta-1}{\beta-\gamma}} dx = \infty,$$

then the equation (4.5) with $\alpha = 1$, $a(x) \equiv 1$ is nodally oscillatory in Ω .

When we take $\alpha = 1$, $m = 1$, $a(x) \equiv 1$, $C(x) \equiv 0$, Corollaries 4.2–4.3 reduce to Corollaries 3–4 given in [16], respectively.

Inequality (2.8) is utilized to establish Wirtinger-type inequality concerning the elliptic type nonlinear equation $P(v) = 0$. We know that a typical Wirtinger inequality is the following.

Theorem 4.4 ([14]). If $u(t) \in C^1([a, b])$ and $u(a) = u(b) = 0$ then

$$\int_a^b u'^2(t) dt \geq \left(\frac{\pi}{b-a} \right)^2 \int_a^b u^2(t) dt$$

where equality holds if and only if

$$u(t) = k_0 \sin \frac{\pi(t-a)}{b-a}$$

for some constant k_0 .

Using Theorem 3.3, the following Wirtinger-type inequality can be easily obtained.

Theorem 4.5. Let $\partial G \in C^1$. Assume that there exists a solution v of $\mathcal{D}_P(G)$ of $P(v) = 0$ such that $v \neq 0$ in \bar{G} . If $u \in C^1(\bar{G}, R)$ and $u = 0$ on ∂G , then

$$\int_G A(x) \left| \nabla u - \frac{uB(x)}{A(x)} \right|^{\alpha+1} dx \geq \int_G C_1(x) |u|^{\alpha+1} dx. \quad (4.7)$$

Remark 4.6. Note that when we take $B(x) \equiv 0$, we have $0 \leq M(u) = M(c_0 v) = 0$, we observe that $M(u) = 0$. When $B(x) \equiv 0$, $D_i(x) \equiv E_j(x) \equiv 0$, ($i = 1, 2, \dots, \ell$; $j = 1, 2, \dots, m$), Theorem 4.5 gives Corollary 4.2 in [34].

References

- [1] R. A. ADAMS, J. J. F. FOURNIER, *Sobolev spaces*, 2nd edition, Pure and Applied Mathematics (Amsterdam), Vol. 140, Academic Press, 2003. [MR2424078](#)
- [2] N. AHMED, D. K. SUNADA, Nonlinear flow in porous media, *J. Hydraulics Division Proc. Amer. Soc. Civil Eng.* **95**(1969), 1847–1857.
- [3] W. ALLEGRETTO, Sturm theorems for degenerate elliptic equations, *Proc. Amer. Math. Soc.* **129**(2001), 3031–3035. [MR1840109](#)
- [4] W. ALLEGRETTO, Y. X. HUANG, A Picone's identity for the p -Laplacian and applications, *Nonlinear Anal.*, **32**(1998), 819–830. [MR1618334](#)
- [5] W. ALLEGRETTO, Y. X. HUANG, Principal eigenvalues and Sturm comparison via Picone's identity, *J. Differential Equations* **156**(1999), 427–438. [MR1705379](#)
- [6] A. AMBROSETTI, A. MALCHIODI, *Nonlinear analysis and semilinear elliptic problems*, Cambridge University Press, Cambridge, 2007. [MR2292344](#)
- [7] R. ARIS, *The mathematical theory of diffusion and reaction in permeable catalysts. Vols. I and II*, Clarendon Press, Oxford, 1975.
- [8] G. ASTARITA, G. MARRUCI, *Principles of non-Newtonian fluid mechanics*, McGraw-Hill, New York, 1974.
- [9] G. BOGNÁR, O. DOŠLÝ, The application of Picone-type identity for some nonlinear elliptic differential equations, *Acta Math. Univ. Comenian.* **72**(2003), 45–57. [MR2020577](#)
- [10] C. CLARK, C. A. SWANSON Comparison theorems for elliptic differential equations, *Proc. Amer. Math. Soc.* **16**(1965), 886–890. [MR0180753](#)

- [11] J. I. DIAZ, *Nonlinear partial differential equations and free boundaries. Vol. I. Elliptic equations*, Research Notes in Mathematics, Vol. 106, Pitman, London, 1985. [MR0853732](#)
- [12] D. R. DUNNINGER, A Sturm comparison theorem for some degenerate quasilinear elliptic operators, *Boll. Un. Mat. Ital. A (7)* **9**(1995), 117–121. [MR1324611](#)
- [13] L. C. EVANS, *Partial differential equations*, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, Providence, RI, 1998. [MR1625845](#)
- [14] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, Cambridge Univ. Press, 1988.
- [15] J. JAROŠ AND T. KUSANO, A Picone type identity for second order half-linear differential equations, *Acta Marth. Univ. Comenian.* **68**(1999), 137–151. [MR1711081](#)
- [16] J. JAROŠ, T. KUSANO, N. YOSHIDA, Picone-type inequalities for nonlinear elliptic equations and their applications, *J. Inequal. Appl.* **6**(2001), 387–404. [MR1888432](#)
- [17] J. JAROŠ, T. KUSANO, N. YOSHIDA, Picone-type inequalities for elliptic equations with first order terms and their applications, *J. Inequal. Appl.* **2006**, Art. ID 52378, 1–17. [MR2215467](#)
- [18] K. KREITH, A strong comparison theorem for self adjoint elliptic equations, *Proc. Amer. Math. Soc.* **19**(1968) 989–990.
- [19] K. KREITH, C. C. TRAVIS, Oscillation criteria for selfadjoint elliptic equations, *Pacific J. Math.*, **41**(1972) 743–753. [MR0318642](#)
- [20] K. KREITH, *Oscillation Theory*, Lecture Notes in Mathematics, Vol. 324, Springer-Verlag, Berlin, 1973.
- [21] K. KREITH, Picone’s identity and generalizations, *Rend. Mat. (6)* **8**(1975), 251–261. [MR0374561](#)
- [22] T. KUSANO, J. JAROŠ, N. YOSHIDA, A Picone-type identity and Sturmian comparison and oscillation theorems for a class of half-linear partial differential equations of second order, *Nonlinear Anal., Ser A: Theory Methods* **40**(2000), 381–395. [MR1768900](#)
- [23] J. T. ODEN, *Existence theorems and approximations in nonlinear elasticity*, in: E. Lakshmikantham (Ed.), *Nonlinear Equations in Abstract Space*, North-Holland, 1978.
- [24] M.-C. PÉLISSIER, *Sur quelques problèmes non linéaires en glaciologie (in French) [On some nonlinear problems of glaciology]*, U. E. R. Mathématique, Université Paris XI, Orsay, 1975. [MR0439015](#)
- [25] M. PICONE, Sui valori eccezionali di un parametro da cui dipende un’equazione differenziale lineare ordinaria del second’ordine (in Italian) [On exceptional values of a parameter which depends on the linear ordinary differential equation of second order], *Ann. Scuola Norm. Sup. Pisa* **11**(1910), 144 pp. [MR1556637](#)
- [26] M. SCHOENAUER, A monodimensional model for fracturing, in: A. Fasano, M. Primicerio (editors), *Free Boundary Problems: Theory, Applications*, Res. Notes Math., Vol. 79, Pitman, London, 1983, 701–711. [MR714899](#)
- [27] C. STURM, Sur les équations différentielles linéaires du second ordre (in French) [On second order linear differential equations], *J. Math. Pures. Appl.* **1**(1836), 106–186.

- [28] C. A. SWANSON, A comparison theorem for elliptic differential equations, *Proc. Amer. Math. Soc.* **17**(1966), 611–616. [MR0201781](#)
- [29] C. A. SWANSON, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968. [MR0463570](#)
- [30] N. YOSHIDA, Oscillation of half-linear partial differential equations with first order terms, *Stud. Univ. Žilina Math. Ser.* **17**(2003), 177–184. [MR2064993](#)
- [31] N. YOSHIDA, Oscillation criteria for half-linear partial differential equations via Picone's identity, in: *Proceedings of Equadiff-11*, 2005, 589–598.
- [32] N. YOSHIDA, *Oscillation theory of partial differential equations*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008. [MR2485076](#)
- [33] N. YOSHIDA, Sturmian comparison and oscillation theorems for a class of half-linear elliptic equations, *Nonlinear Anal.* **71**(2009), e1354–e1359. [MR2671922](#)
- [34] N. YOSHIDA, A Picone identity for half-linear elliptic equations and its applications to oscillatory theory, *Nonlinear Anal.* **71**(2009), 4935–4951.
- [35] N. YOSHIDA, Sturmian comparison and oscillation theorems for quasilinear elliptic equations with mixed nonlinearities via Picone-type inequality, *Toyama Math. J.* **33**(2010), 21–41. [MR2893644](#)